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BUCKLING OF AN ECCENTRICALLY COMPRESSED BAR

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The problem of determining the shape of the elastic line of a longitudinally compressed bar has an exact solution in elliptic functions [1–3] and hence, besides being valuable by itself, is exceedingly useful as a test problem in constructing effective approximate solutions. In the present work the postbuckling behavior of an ideally loaded bar is studied by the perturbation method. The approximate formulas obtained are used to analyze buckling by the method of successive additional loading of an eccentrically compressed bar (a nonideal system), and in so doing, in every step of additional loading use is made of the information on the distortion of the critical load spectrum of a buckled ideal system. A comparison of the results shows the applicability limitations of the approximate formulas: calculations using these formulas do not require any tables of elliptic integrals and are readily made on calculators.

1. An Ideally Loaded Bar. Analysis of Buckling by the Perturbation Method. We consider a hinge-supported bar of length L (Fig. 1) loaded by an axial compressive force P, which retains its magnitude and direction upon deformation of the bar. We assume that the length L of the bar's axial line is invariable, and the bar axis can bend only in the plane (x, y). Let us investigate the buckling mode and the postbuckling behavior using the perturbation technique.

Let us use the exact equation of elastic equilibrium upon plane deflection of the bar in the form [1–5]

$$æ = M/EI, \tag{1.1}$$

where æ is the curvature at a given point of the elastically deflected longitudinal bar axis; M is the bending moment at this point; and EI is the flexural rigidity. We express the bending moment $M = -Pw$ and the curvature $æ = w_{ss}/(1 - w_s^2)^{1/2}$ at an arbitrary point s of the bar via the function $w(s)$, which completely [6] determines the strained state of the bar (Fig. 1). We substitute these expressions into Eq. (1.1) and differentiate it twice with respect to s:

$$(w_{ss}/(1 - w_s^2)^{1/2})_{ss} = -Pw_{ss}/EI.$$

Thus, the function $w(s)$ satisfies the differential equation

$$w_{ssss} + \left[\frac{P}{EI} + \frac{w_{ss}^2(1 + 2w_s^2) + 3w_s w_{sss}(1 - w_s^2)}{(1 - w_s^2)^{5/2}}\right](1 - w_s^2)^{1/2} w_{ss} = 0 \tag{1.2}$$

and the boundary conditions

$$w(0) = w(L) = w_{ss}(0) = w_{ss}(L) = 0.$$

We write Eq. (1.2) with accuracy up to sixth-power terms in the function $w(s)$ and its derivatives, inclusive:

$$w_{ssss} + \left[w_{ss}^2(1 + 4w_s^2) + 3w_s w_{sss}(1 + w_s^2)\right] w_{ss} + \frac{P}{EI}\left(1 - \frac{1}{2}w_s^2 - \frac{1}{8}w_s^4\right) w_{ss} = 0. \tag{1.3}$$

We introduce a new variable $z = \pi s/L$ and a function W such that $w = \alpha W$ (α is a constant of the same order of smallness as the deflection amplitude). With this notation Eq. (1.3), takes the form

$$\alpha\left(\frac{\pi}{L}\right)^4 W_{zzzz} + \left[\alpha^2\left(\frac{\pi}{L}\right)^4 W_{zz}^2\left(1 + 4\alpha^2\left(\frac{\pi}{L}\right)^2 W_z^2\right) + 3\alpha^2\left(\frac{\pi}{L}\right)^4 W_z W_{zzz}\left(1 + \alpha^2\left(\frac{\pi}{L}\right)^2 W_z^2\right)\right]\alpha\left(\frac{\pi}{L}\right)^2 W_{zz}$$

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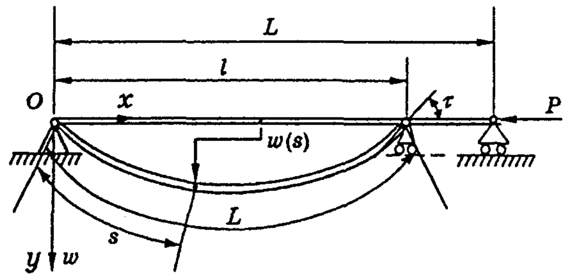

Fig. 1

$$+\frac{P}{EI}\left[1-\frac{1}{2}\alpha^2\left(\frac{\pi}{L}\right)^2 W_z^2-\frac{1}{8}\alpha^4\left(\frac{\pi}{L}\right)^4 W_z^4\right]\alpha\left(\frac{\pi}{L}\right)^2 W_{zz}=0. \tag{1.4}$$

Dividing Eq. (1.4) by $\alpha/(\pi/L)^4$ and denoting by $\varepsilon=\pi^2(\alpha/L)^2$ the small parameter characterizing the deviation of the system from the trivial solution $W=0$, we obtain

$$W_{zzzz}+\varepsilon\left[W_{zz}^2\left(1+4\varepsilon W_z^2\right)+3W_zW_{zzz}\left(1+\varepsilon W_z^2\right)\right]W_{zz}+\frac{P}{EI}\left(\frac{L}{\pi}\right)^2\left[1-\frac{1}{2}\varepsilon W_z^2-\frac{1}{8}\varepsilon^2W_z^4\right]W_{zz}=0.$$

Thus, with accuracy up to ε^2 we have the following problem on eigenfunctions and eigenvalues:

$$(A_0+\varepsilon A_1+\varepsilon^2A_2)\,W-\Lambda\,(B_0+\varepsilon B_1+\varepsilon^2B_2)\,W=0,\quad W(0)=W(\pi)=W_{zz}(0)=W_{zz}(\pi)=0. \tag{1.5}$$

Here $\Lambda=P\,(L/\pi)^2/EI$; A_0 and B_0, and A_1, B_1, A_2, and B_2 are the linear and nonlinear operators

$$A_o=(\,)_{zzzz},\qquad B_0=-(\,)_{zz},$$

$$A_1=(\,)_{zz}^3+3(\,)_z(\,)_{zz}(\,)_{zzz},\qquad B_1=\frac{1}{2}(\,)_z^2(\,)_{zz},$$

$$A_2=4(\,)_z^2(\,)_{zz}^3+3(\,)_z^3(\,)_{zz}(\,)_{zzz},\qquad B_2=\frac{1}{8}(\,)_z^4(\,)_{zz}.$$

For any Λ the trivial solution $W\equiv 0$ of problem (1.5) relates to the unbuckled equilibrium state of the bar. Problem (1.5) at $\varepsilon=0$ is called an unperturbed (linearized) problem. The eigenfunctions of the unperturbed problem are $W_n^{(0)}=\gamma_n\sin nz$ (γ_n are the normalization coefficients) and the eigenvalues are $\Lambda_n^{(0)}=n^2$. Following the conventional procedures of perturbation theory [7], we represent the eigenfunctions W_n and eigenvalues Λ_n of the perturbed problem (1.5) in the form of asymptotic series in terms of the parameter ε:

$$W_n=W_n^{(0)}+\sum_{k=1}^{\infty}\varepsilon^kW_n^{(k)},\quad \Lambda_n=\Lambda_n^{(0)}+\sum_{k=1}^{\infty}\varepsilon^k\Lambda_n^{(k)}. \tag{1.6}$$

The normalization conditions for the unperturbed problem are given by the relationship $(B_0W_i^{(0)},W_j^{(0)})=\delta_{ij}$ (δ_{ij} are the Kronecker symbols), from which the equality $n^2\gamma_n^2=2/\pi$ for determining the coefficients γ_n follows. For the perturbed problem (1.5), these conditions are specified, in addition, by the relationship $(W_i^{(0)},W_i^{(1)})=0$. Here by the scalar product is meant the functional

$$(f(z),g(z))=\int_0^{\pi}fg\,dz.$$

We substitute the asymptotic expansions (1.6) into the equation and boundary conditions of problem (1.5) and equate to zero the coefficients of terms with the same powers ε. Using the normalization conditions, we find the expansions of Λ_n with accuracy up to ε^2 and the expansions of W_n with accuracy up to first-power terms in ε inclusive. Equating to zero the coefficient of ε, we obtain the equation

$$A_0W_n^{(1)}+A_1W_n^{(0)}-\Lambda_n^{(0)}\,(B_0W_n^{(1)}+B_1W_n^{(0)})-\Lambda_n^{(1)}B_0W_n^{(0)}=0, \tag{1.7}$$

which contains, as unknowns, the functions $W_n^{(1)}(z)$ and the parameters $\Lambda_n^{(1)}$. Substituting into (1.7) the expansion of the function $W_n^{(1)}$ into the series

$$W_n^{(1)} = \sum_{j=1}^{\infty} \alpha_{nj} W_j^{(0)}$$

in terms of the eigenfunctions of the unperturbed equation and using the normalization conditions, we have

$$\Lambda_n^{(1)} = (A_1 W_n^{(0)}, W_n^{(0)}) - \Lambda_n^{(0)} (B_1 W_n^{(0)}, W_n^{(0)}), \quad \alpha_{nn} = 0,$$

$$\alpha_{nj} = [\Lambda_n^{(0)} (B_1 W_n^{(0)}, W_j^{(0)}) - (A_1 W_n^{(0)}, W_j^{(0)})]/(\Lambda_j^{(0)} - \Lambda_n^{(0)}), \quad j \neq n.$$

Calculating these quantities, we find the eigenfunctions and eigenvalues of the perturbed problem (1.5) with accuracy up to first-power terms in ε:

$$W_n(z) = \gamma_n \sin nz - \varepsilon \frac{3}{32\pi} \gamma_{3n} \sin 3nz, \quad \Lambda_n = n^2 + \varepsilon \gamma_n^2 n^4/8.$$

Substituting expansions (1.6) into the equation of problem (1.5), equating to zero the coefficient of ε^2, etc., and following the procedures of the perturbation theory, we find the expressions for Λ_n and $W_n(z)$ with accuracy up to terms containing ε^2 inclusive.

In the nonlinear problem (1.5), of greatest practical interest are the least eigenvalue

$$\Lambda_1 = 1 + \frac{1}{8} \gamma_1^2 \varepsilon + \frac{21}{512} \gamma_1^4 \varepsilon^2$$

and the eigenfunction

$$W_1(z) = \gamma_1 \sin z - \varepsilon \frac{3}{32\pi} \gamma_3 \sin 3z$$

corresponding to it with accuracy up to ε. Returning to the initial notation ($z = \pi s/L$, $w = \alpha W$, $\varepsilon = \pi^2 (\alpha/L)^2$, $\Lambda = P(L/\pi)^2/EI$), we obtain

$$P = EI\left(\frac{\pi}{L}\right)^2 \left[1 + \frac{\pi^2}{8}\left(\frac{\alpha\gamma_1}{L}\right)^2 + \frac{21\pi^4}{512}\left(\frac{\alpha\gamma_1}{L}\right)^4\right] \tag{1.8}$$

and, taking into account the equation $(3\gamma_3)^2 = \gamma_1^2 = 2/\pi$, which follows from the normalization conditions, we write

$$w_1(s) = \alpha\gamma_1 \left[\sin\frac{\pi s}{L} - \frac{\pi^2}{64}\left(\frac{\alpha\gamma_1}{L}\right)^2 \sin\frac{3\pi s}{L}\right]. \tag{1.9}$$

Formula (1.9) gives an approximate expression for the coordinate $w_1(s)$ of the curved bar axis. The maximal deflection of the bar is reached at point $s = L/2$ and has the form

$$f_1 = \max_{0 \leqslant s \leqslant L} w_1(s) = \alpha\gamma_1 \left[1 + \frac{\pi^2}{64}\left(\frac{\alpha\gamma_1}{L}\right)^2\right]. \tag{1.10}$$

From (1.8) and (1.10), with accuracy up to $(f_1/L)^4$ inclusive, we obtain the dependence

$$P = EI\left(\frac{\pi}{L}\right)^2 \left[1 + \frac{\pi^2}{8}\left(\frac{f_1}{L}\right)^2 + \frac{19\pi^4}{512}\left(\frac{f_1}{L}\right)^4\right], \tag{1.11}$$

which binds the external load P with the quantity f_1 or, solving the latter equality with respect to f_1, we obtain

$$f_1 = \frac{4\sqrt{2}}{\pi} L \sqrt{\left(\sqrt{1 + 19(P/P_* - 1)/2} - 1\right)/19} \tag{1.12}$$

$[P_* = EI(\pi/L)^2]$. At $t = 19(P/P_* - 1)/2 \ll 1$ equality (1.12) can be written with accuracy up to second-order

TABLE 1

τ, deg	Load P/P_*	Deflection f_1/L		
		Exact value	Formula (1.12)	Formula (1.13)
10	1.0038	0.0554	0.0554	0.0554
20	1.0154	0.1097	0.1098	0.1096
30	1.0351	0.1620	0.1625	0.1617
40	1.0637	0.2111	0.2134	0.2100
60	1.1517	0.2966	0.3098	0.2875
70	1.2147	0.3313	0.3562	0.3108
80	1.2939	0.3597	0.4021	0.3177

infinitesimals in t as

$$f_1 = \frac{2\sqrt{2}}{\pi} L\sqrt{P/P_* - 1}\,[1 - 19(P/P_* - 1)/16], \tag{1.13}$$

which is more accurate than the equation [1, p. 74]

$$f_1 = \frac{2\sqrt{2}}{\pi} L\sqrt{P/P_* - 1}\left[1 - (P/P_* - 1)/8\right]. \tag{1.14}$$

The results of calculations using formulas (1.12) and (1.13) are presented in Table 1. Moreover, this table gives, for different values of the angle τ between the tangent of the bar axis at its vertex and the Ox axis (Fig. 1), the exact values of deflections (see [1, p. 73; 2, p. 509; 3, p. 74]) calculated, with the aid of the tables of elliptic integrals, from the formulas

$$F = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}, \quad \frac{P}{P_*} = \left(\frac{2F}{\pi}\right)^2, \quad \frac{f_1}{L} = \frac{k}{F} \tag{1.15}$$

$[k = \sin(\tau/2)]$. A comparison of the approximate values of the deflections calculated from (1.12) with the exact values shows a coincidence, technically quite acceptable, up to loads exceeding the critical load by a factor of 1.15. Note that, in calculating the bar deflection using formula (1.14), for $\tau = 30°$ we have $f_1/L \approx 0.168$ instead of the exact value $f_1/L = 0.162$, and $f_1/L \approx 0.406$ for $\tau = 70°$ instead of $f_1/L = 0.3313$.

Let us show that formula (1.11) can be obtained from (1.15) by expanding P/P_* into a series in terms of the powers of f_1/L. To do so, we substitute into the second relationship of (1.15) the representation

$$F = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots\right) \tag{1.16}$$

of the complete elliptic first-order integral in the form of a series [7] converging for $|k| < 1$ and obtain

$$\frac{P}{P_*} = 1 + \frac{1}{2}k^2 + \frac{11}{32}k^4 + \ldots\,. \tag{1.17}$$

Replacing k^2 in (1.17) by the expression $(\pi^2/4)(P/P_*)(f_1/L)^2$, which follows from the second and third relationships of (1.15), and taking into account the absolute convergence of series (1.16), we have the expression

$$\frac{P}{P_*} = 1 + \frac{1}{2}\frac{\pi^2}{4}\left[\frac{P}{P_*}\right]\left(\frac{f_1}{L}\right)^2 + \frac{11}{32}\left\{\frac{\pi^2}{4}\left[\frac{P}{P_*}\right]\left(\frac{f_1}{L}\right)^2\right\}^2 + \ldots$$

$$= 1 + \frac{\pi^2}{8}\left[1 + \frac{1}{2}\frac{\pi^2}{4}\left(\frac{f_1}{L}\right)^2 + \ldots\right]\left(\frac{f_1}{L}\right)^2 + \frac{11}{32}\left\{\frac{\pi^2}{4}\left(\frac{f_1}{L}\right)^2\right\}^2 + \ldots = 1 + \frac{\pi^2}{8}\left(\frac{f_1}{L}\right)^2 + \frac{19\pi^4}{512}\left(\frac{f_1}{L}\right)^4 + \ldots, \tag{1.18}$$

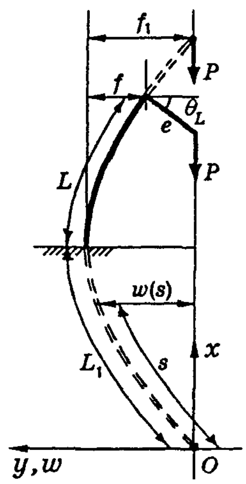

Fig. 2

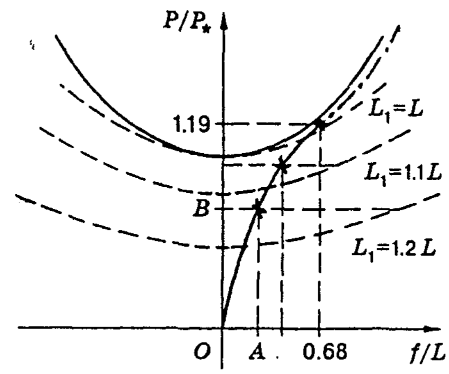

Fig. 3

which coincides with (1.11) with accuracy up to terms containing $(f_1/L)^4$. From the second and third relationships of (1.15) and the obvious inequality $f_1 < L/2$ follows the inequality $k^2 = (\pi^2/4)(P/P_*)(f_1/L)^2 < (\pi^2/16)(P/P_*) < 1$, if $P/P_* < 16/\pi^2 \approx 1.62$. This inequality is sufficient for the convergence of series (1.16) and (1.17). Consequently, using partial sums of series (1.18), we can obtain for $P/P_* < 1.62$ the load–deflection dependence to any desired accuracy.

2. Analysis of Buckling of an Eccentrically Compressed Bar by the Method of Additional Loading. Figure 2 shows a constraint bar of length L deflected under the action of force P which retains its vertical direction and is applied with eccentricity e [4]. Also shown here is a fictitious hinge-supported bar of length $2L_1$ whose ends are lying in the line of action of force P. For the deflection function $w(s)$ $(0 \leqslant s \leqslant L)$ of the fictitious bar use can be made of the approximate expression (1.9) by substituting in it $2L_1$ for L. The slope angle θ_L of the top end is expressed via the deflection function [4, 5]

$$\cos\theta_L = (1 - w_s^2(L_1 + L))^{1/2}.$$

The maximal deflection of the bar is denoted by f and that of the fictitious bar, by f_1. Since $f_1 - f = e\cos\theta_L$ (Fig. 2),

$$w(L_1 + L) = e(1 - w_s^2(L_1 + L))^{1/2}$$

and hence, substituting expression (1.9) for w, we find the equation

$$(\alpha\gamma_1)^2\left[\sin\frac{\pi(L_1+L)}{2L_1} - \frac{\pi^2}{64}\left(\frac{\alpha\gamma_1}{L}\right)^2\sin\frac{3\pi(L_1+L)}{2L_1}\right]^2$$
$$= e^2\left\{1 - (\alpha\gamma_1)^2\left[\frac{\pi}{2L_1}\cos\frac{\pi(L_1+L)}{2L_1} - \frac{\pi^2}{64}\left(\frac{\alpha\gamma_1}{L}\right)^2\frac{3\pi}{2L_1}\cos\frac{3\pi(L_1+L)}{2L_1}\right]^2\right\},$$

which makes it possible to determine $\alpha\gamma_1$ from the given L, L_1, and e. By using the notation

$$\mu = \left(\frac{\alpha\gamma_1\pi}{2L_1}\right)^2, \quad b = \left(\frac{e\pi}{2L_1}\right)^2, \quad c = \sin^2\left(\frac{L\pi}{2L_1}\right),$$

we obtain, with accuracy up to $(\alpha\gamma_1)^4$ inclusive, the equation

$$[4(1-3b)c^2 - (5-9b)c + 1]\mu^2/32 + [(b-1)c + 1]\mu - b = 0,$$

which is quadratic in μ and whose discriminant is

$$D = (1-c)^2 + b(1 + 11c - 12c^2)/8 + b^2c(9-4c)/8 \geqslant 5b^2c/8 \geqslant 0.$$

TABLE 2

L_1	Approximate solution		Exact solution	
	P/P_*	f/L	P/P_*	f/L
1.2	0.7140	0.2615	0.7141	0.2618
1.1	0.8913	0.4441	0.8946	0.4472
1.06	1.0032	0.5589	1.0209	0.5707
1.03	1.1044	0.6423	1.1683	0.6750
1.02	1.1375	0.6630	1.2330	0.7076
1.01	1.1679	0.6775	1.3076	0.7372
1	1.1942	0.6846	1.3932	0.7628

TABLE 3

e	Approximate solution		Exact solution	
	P/P_*	f/L	P/P_*	f/L
0.1	0.89129	0.44407	0.89461	0.44722
0.01	0.82748	0.06001	0.82741	0.05999
0.001	0.82646	0.00603	0.82657	0.00604

Finally, for $\alpha\gamma_1$, we have

$$\alpha\gamma_1 = \frac{8L_1}{\pi}\sqrt{\frac{1+(b-1)\,c-\sqrt{D}}{4\,(3b-1)\,c^2+(5-9b)\,c-1}}, \tag{2.1}$$

which is readily calculated for the given e, L, and L_1. The choice of the sign before $D^{1/2}$ was determined by the physical meaning of the parameter $\alpha\gamma_1$, which is approximately equal to the bar deflection amplitude. Thus, if before $D^{1/2}$ we put the plus sign, then at $L_1/L \geqslant 4$ the value of the fraction under the radical becomes negative, and at $L_1/L = 2$ we obtain $\alpha\gamma_1 \approx 7L$.

Now we will present the construction algorithm for a load–deflection diagram. First we assume that $L_1 = 3L$ and, in a certain step ($0.1L$ or $0.01L$), we begin to decrease L_1 to L. In other words, we will successively apply additional loads to the bar and increase the load P until the slope angle of the top end of the bar becomes equal to $\pi/2$. For each loading step, i.e., for each L_1, we calculate $\alpha\gamma_1$ from (2.1).

Load–deflection diagrams for an ideal bar of length L_1, which were constructed by the approximate formulas (1.10) and (1.11), are shown by the dashed lines in Fig. 3. Substituting the value of $\alpha\gamma_1$ into expression (1.9) of the deflection function for an ideal system, we can calculate approximately the horizontal shift of $w\,(s)$ at any point s of the bar (see Fig. 2). Thus, for the upper end of the real bar, we have $s = L_1 + L$, hence the maximal deflection is $f = w\,(L_1) - w\,(L_1 + L)$ (point A in Fig. 3), and the load corresponding to it (point B in Fig.3) can be calculated from formula (1.8) or (1.11) by substituting in them $2L_1$ for L. Thereby, for the given L and L_1, we can approximately calculate the bar deflection and the load. Assuming various values of L_1, we construct points (marked in Fig. 3 by a cross) of the load-deflection curve of an eccentrically compressed bar.

Table 2 lists the calculation results obtained by the above algorithm for $e = 0.1$, $L = 1$. By decreasing L_1 successively from 1.2 to 1 (via additional loading of the bar), we obtained a series of points of the load–deflection diagram. Also shown here are the refined values calculated by linear interpolation of the table data [3] in solving the system of equations (1.200), (1.205), and (1.206) [4, pp. 69, 70] with three unknowns, a part of which is under an incomplete elliptic first-order integral. Table 2 shows good agreement of the results up to $L_1/L = 1.03$, i.e., up to loads exceeding the critical load by 10%. However, if we assume that P/P_* rather than L_1 is known, the difference of the deflections obtained by the approximate method from the exact deflections becomes insignificant (see the approximate value of deflection for $L_1 = 1.01$, and the exact one for $L_1 = 1.03$, in Table 2) within the entire range of applicability of the method, i.e., up to $P/P_* = 1.19$. Thus, for $P/P_* = 1.1044$, we have $f/L \approx 0.6423$ instead of the exact $f/L = 0.6353$, and for $P/P_* = 1.1942$, we have $f/L \approx 0.6846$ instead of 0.6889, i.e., the difference of the approximate values of the deflections obtained by the method described from the exact values for the same external load is about 1%. Thus, the proposed method makes it possible to accurately construct the load–deflection curve up to loads exceeding the critical load by 19% at which the bar deflection reaches ≈ 0.68 of its length. The dash-and-dot curve in Fig. 3 is not reproduced using this method.

From Table 3, which presents, for $L_1 = 1, 1L$, the effect of eccentricity on the error of approximate results, it can be seen that as the eccentricity decreases to $e = 0.001$ the error does not exceed 0.7%. The effect of eccentricity on the discrepancy between the curves corresponding to the exact load–deflection dependence at $P > P_*$ drops [4, p. 70, Fig. 1.29]. This is as it should be, because the potential function of an ideal bar has the form of a cusp catastrophe [8], and hence the stability along the equilibrium curves of a bar with small initial imperfection (for example, upon eccentrical loading) is completely determined by the stability properties of the cusp catastrophe.

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